Eigenvalue distributions and Weyl laws for semi-classical non-self-adjoint operators in 2 dimensions

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Dedicated to Hans Duistermaat

Abstract

In this note we compare two recent results about the distribution of eigenvalues for semi-classical pseudodifferential operators in two dimensions. For classes of analytic operators A. Melin and the author [6] obtained a complex Bohr-Sommerfeld rule, showing that the eigenvalues are situated on a distorted lattice. On the other hand, with M. Hager [4] we showed in any dimension that Weyl asymptotics holds with probability close to 1 for small random perturbations of the operator. In both cases the eigenvalues distribute to leading order according two smooth densities and we show here that the two densities are in general different.

Résumé

Dans cette note nous comparons deux résultats récents sur la distribution des valeurs propres d'opérateurs pseudodifférentiels en deux dimensions. Pour des classes d'opérateurs analytiques A. Melin et l'auteur [6] a obtenu une loi de Bohr-Sommerfeld complexe qui montre que les valeurs propres sont situés sur un réseau déformé. D'autre part, avec M. Hager [4] nous avons montré dans toute dimension que l'asymptotique de Weyl est valable avec probabilité proche de 1 pour des petites perturbations aléatoires de l'opérateur. Dans les deux cas, le valeurs propres sont distribuées (à des petites corrections près) selon des densités lisses, et ici nous montrons que ces densités sont en général différentes.

1 Introduction

In the classical paper by J.J. Duistermaat and L. Hörmander [1], one very interesting application is about (pseudo)differential operators with principal symbol p such that the Poisson bracket $\{p, \overline{p}\}$ vanishes on the zero set $p^{-1}(0)$ and the differentials of the real and imaginary part of p are independent there, so that the zero set is a codimension 2 sub-manifold of the cotangent space. The authors gave interesting existence results under non-compactness assumptions on the bicharacteristic foliation. In my thesis under the direction of L. Hörmander my task was to study the case when $\{p, \overline{p}\} \neq 0$ on the zero-set of the symbol and in a subsequent paper with Duistermaat [2] we introduced an studied certain microlocal projections onto the kernel and the co-kernel of the operator. The full history of this subject can be traced back to the famous counterexample of Hans Lewy to local solvability and subsequent works by Hörmander and others and there is also quite a rich recent history.

There has been a renewed interest in non-self-adjoint operators and the related notion of pseudospectrum, promoted by L.N. Trefethen, E.B. Davies, M. Zworski and others. Again the Poisson bracket $i^{-1}\{p, \overline{p}\}$ plays an important role as a source of pseudospectral behaviour, including spectral instability. (We observe here that the above Poisson bracket is equal to the principal symbol of the commutator of the corresponding (pseudo)differential operator and its adjoint.) We refer to the surveys [8, 9] where further references can be found.

Possibly, as a reaction to these developments, the author participated in two projects:

- With A. Melin [6] we discovered for a a fairly wide and stable class of non-self-adjoint semi-classical pseudodifferential operators with analytic symbols that the individual eigenvalues in certain regions can be determined by a Bohr-Sommerfeld quantization rule defined in terms of certain complex Lagrangian tori (close to the real domain). The underlying idea is here to change the Hilbert space norm by means of exponential weights in such a way that the operator becomes (more) normal.

- M. Hager [3] considered certain non-self-adjoint h-pseudodifferential operators in dimension 1 with small multiplicative random perturbations and showed that with probability tending to 1 when $h \to 0$, the eigenvalues distribute according to the classical Weyl law, well-known in the context of self-adjoint operators since almost a century. The same type of result was subsequently obtained in any dimension by Hager

and the author [4] for a certain class of non-multiplicative random perturbations and recently also for multiplicative random perturbations in any dimension by the author [10].

In the present note we shall compare the resulting distributions of eigenvalues, in the case when the unperturbed operator satisfies the assumptions of [6], and we shall see that they are in general different. This means that the random perturbations will change radically the asymptotic distribution of eigenvalues. The intuitive explanation of this phenomenon is that the result of [6] depends on the geometry in the complex domain, while the random perturbation destroys analyticity and hence the eigenvalue distribution should be given in terms of the real phase space, where the Weyl law is the natural candidate.

We next describe the main result of [6]. Let $p(x,\xi)$ be bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^4$ in $\mathbf{C}^4 = \mathbf{C}^2_x \times \mathbf{C}^2_\xi$. (The assumptions near ∞ can be varied in many ways and we can let p belong to some more general symbol space as long as we have the appropriate form of ellipticity near infinity, cf (1.2) below.) Assume that

$$\mathbf{R}^4 \cap p^{-1}(0) \neq \emptyset \text{ is connected,} \tag{1.1}$$

for simplicity. Also assume that

$$\text{on } \mathbf{R}^4 \text{ we have } |p(x,\xi)| \geq 1/C, \text{ for } |(x,\xi)| \geq C, \tag{1.2}$$

for some $C > 0$,

$$d\Re p(x,\xi), d\Im p(x,\xi) \text{ are linearly independent for all } (x,\xi) \in p^{-1}(0) \cap \mathbf{R}^4. \tag{1.3}$$

It follows that $p^{-1}(0) \cap \mathbf{R}^4$ is a compact (2-dimensional) surface.

Also assume that

$$|\{\Re p, \Im p\}| \text{ is sufficiently small on } p^{-1}(0) \cap \mathbf{R}^4. \tag{1.4}$$

By "sufficiently small", we mean that $|\{\Re p, \Im p\}| < \delta$ for some $\delta > 0$ that will depend on all constants (implicit or explicit) that are required to express the other conditions above uniformly.

In [6] we showed that $p^{-1}(z) \cap \mathbf{R}^4$ is a real torus for $z \in \text{neigh}\,(0, \mathbf{C})$ (ie some neighborhood of 0 in $\mathbf{C}$) and that there exists a smooth 2-dimensional torus $\Gamma(z) \subset p^{-1}(z) \cap \mathbf{C}^4$, close to $p^{-1}(z) \cap \mathbf{R}^4$ such that $\sigma_{|\Gamma(z)} = 0$ and $I_j(z) \in \mathbf{R}$, $j = 1, 2$. Here $I_j(z) := \int_{\gamma_j(z)} \xi \cdot dx$ are the actions along two fundamental cycles $\gamma_1(z), \gamma_2(z) \subset \Gamma(z)$ and $\sigma = \sum_1^2 d\xi_j \wedge dx_j$ is the complex symplectic $(2,0)$ form. Moreover, $\Gamma(z)$, $I_j(z)$ depend smoothly on $z \in \text{neigh}\,(0)$.

The main result of [6], valid under slightly more general assumptions than the ones above, is then

Theorem 1.1 *Under the above assumptions, there exist a neighborhood V of $0 \in \mathbf{C}$, $\theta_0 \in (\frac{1}{2}\mathbf{Z})^2$, $\theta_j \in C^\infty(V;\mathbf{R}^2)$ and $\theta(z;h) \sim \theta_0 + \theta_1(z)h + \theta_2(z)h^2 + ..$ in $C^\infty(V;\mathbf{R}^2)$, such that for $z \in V$ and for $h > 0$ sufficiently small, z is an eigenvalue of $P = p^w(x, hD_x)$ iff*

$$\frac{(I_1(z), I_2(z))}{2\pi h} = k - \theta(z;h), \text{ for some } k \in \mathbf{Z}^2. \quad (BS)$$

Here $p^w(x, hD)$ denotes the Weyl quantization of the symbol $p(x, h\xi)$.

Let us also assume that

$$\text{the map } z \mapsto I(z) := (I_1(z), I_2(z)) \text{ is a diffeomorphism from } V \text{ to } I(V). \tag{1.5}$$

This assumption is satisfied if we strengthen (1.4) by assuming that $|\{\Re p, \Im p\}|$ is sufficiently small on $p^{-1}(z)$ for all $z \in \operatorname{neigh}(0, \mathbf{C})$ and choose V small enough. The eigenvalues near 0 will then form a distorted lattice and we introduce the leading spectral density function $0 < \omega(z) \in C^\infty(V)$ by

$$dI_1(z) \wedge dI_2(z) = \pm\omega(z) d\Re z \wedge d\Im z, \tag{1.6}$$

where the sign is chosen so that ω becomes positive. Then from Theorem 1.1 it follows that for every $W \Subset V$ with smooth boundary, the number of eigenvalues in W satisfies

$$N(W;h) = \frac{1}{(2\pi h)^2}\left(\int_W \omega(z) L(dz) + o(1)\right), \; h \to 0. \tag{1.7}$$

Here $L(dz) = d\Re z d\Im z$ denotes the Lebesgue measure.

Now we turn to the results in [3, 4, 10]. Again the unperturbed operator is of the form $P = p^w(x, hD_x)$ where the complex-valued smooth symbol should belong to a suitable symbol class and satisfy an ellipticity condition at infinity which guarantees that the spectrum of P in a given open set $\Omega \Subset \mathbf{C}$ is discrete. The perturbed operator is of the form $P_\delta = P + \delta Q_\omega$, where the parameter δ is small, say bounded from above by some positive power of h and from below by $e^{-h^{-\alpha}}$ for some suitable value $\alpha \in]0, 1]$. Under some additional assumptions on the type of random perturbation and about non-constancy of the symbol p, it is showed in the cited works that with a probability that tends to 1 when $h \to 0$, the number of eigenvalues of P_δ in $W \Subset \Omega$ obeys

$$N_\delta(W;h) = \frac{1}{(2\pi h)^n}(\operatorname{vol}(p^{-1}(W)) + o(1)) \tag{1.8}$$

uniformly for W in a class of subsets of Ω with uniformly smooth boundary. (In the case of multiplicative perturbations, an additional symmetry assumption on the symbol is imposed which cannot be completely eliminated.)

Notice that this result can be formulated as in (1.7) with the density ω replaced by the Weyl density $w(z)L(dz)$, defined to be the direct image of the symplectic volume element under the map p, so that

$$\int f(z)w(z)L(dz) = \iint f(p(x,\xi))dxd\xi,\ f \in C_0^\infty(V). \tag{1.9}$$

In the two-dimensional case there are situations (for instance in the case of the symbol $p(x,\xi) = \frac{1}{2}((x_1^2+\xi_1^2)+i(x_2^2+\xi_2^2))-\mathrm{const.}$ and small perturbations of that symbol) where Theorem 1.1 applies to P and the results of [4, 7] apply to small random perturbations, and it is then of interest to compare the spectral densities. We shall see that $\omega(z) = w(z)$ in the integrable case, when $\{\Re p, \Im p\} \equiv 0$ but that these quantities are different in general.

Theorem 1.2 *Under the assumptions (1.1)–(1.5) we have generically that* $w \not\equiv \omega$.

In other words, if $w \equiv \omega$, then there are arbitrarily small perturbations of P within the class of operators as in the theorem, for which $w \not\equiv \omega$.

2 The integrable case

In this section, we strengthen the assumption (1.4) to

$$\{\Re p, \Im p\} \equiv 0. \tag{2.1}$$

It is then well-known by the Liouville-Mineur-Arnold theorem (see [11]) that there exists a real symplectic diffeomorphism $\kappa : \mathrm{neigh}\,(\eta = 0, T^*\mathbf{T}^2) \to \mathrm{neigh}\,(p^{-1}(0) \cap \mathbf{R}^4, \mathbf{R}^4)$, (i.e. from a neighborhood of $\{\eta = 0\}$ in $T^*\mathbf{T}^2$ to a neighborhood of $p^{-1}(0)\cap\mathbf{R}^4$ in $\mathbf{R}^4$) such that

$$p \circ \kappa = \widetilde{p}(\eta) \tag{2.2}$$

is independent of y, where $\mathbf{T}^2 = (\mathbf{R}/2\pi\mathbf{Z})^2$ and $T^*\mathbf{T}^2 \simeq \mathbf{T}^2_y \times \mathbf{R}^2_\eta$.

In this case $\Gamma(z)$ is simply the real Lagrangian torus $p^{-1}(z)\cap\mathbf{R}^4$ and

$$I_j(z) = 2\pi\eta_j + I_j(0), \quad \widetilde{p}(\eta) = z. \tag{2.3}$$

It follows that up to composition with κ, we get the same quantities $\omega(z)$, $w(z)$, if we do the computation directly on $T^*\mathbf{T}^2$ for $\widetilde{p}(\eta)$, and we restrict the attention to that case and drop the tilde.

From (2.3) and (1.6) we get

$$\frac{\omega(z)}{(2\pi)^2} = \left|\det \frac{\partial(\eta_1,\eta_2)}{\partial(\Re p, \Im p)}\right|, \quad p(\eta) = z. \tag{2.4}$$

From (1.9), we get for $f \in C_0^\infty(V)$:

$$\int f(z)w(z)L(dz) = \iint f(p(y,\eta))dyd\eta$$
$$= (2\pi)^2 \int f(p(\eta))d\eta = (2\pi)^2 \int f(z) \left|\det \frac{\partial(\eta_1,\eta_2)}{\partial(\Re p, \Im p)}\right| L(dz),$$

which shows that $w(z)$ also satisfies (2.4), so

$$w(z) = \omega(z),\ z \in V, \tag{2.5}$$

in the completely integrable case (2.1).

3 The general case

In this section we shall prove Theorem 1.2 by means of calculations similar to the ones in [5]. Let p satisfy the assumptions (1.1)–(1.5). Let $G_t(x,\xi)$ for $t \in \operatorname{neigh}(0,\mathbf{R})$ be a smooth family of functions that are holomorphic and uniformly bounded in a fixed tubular neighborhood of $\mathbf{R}^4$. Possibly after decreasing the neighborhood of $t=0$ we get a smooth family of canonical transformations κ_t from a fixed tubular neighborhood of $\mathbf{R}^4$ onto a neighborhood of $\mathbf{R}^4$, by solving the Cauchy problem

$$\frac{d}{dt}\kappa_t(\rho) = (\kappa_t)_*(\widehat{iH_{G_t}})(\rho), \quad \kappa_0(\rho) = \rho, \tag{3.1}$$

where $H_{G_t} = \frac{\partial G_t}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial G_t}{\partial x}\frac{\partial}{\partial \xi}$ is the holomorphic Hamilton field (of type (1,0)) and we identify iH_{G_t} with the corresponding real vector field $\widehat{iH_{G_t}} := iH_{G_t} + \overline{iH_{G_t}}$.

Put $p_t = p \circ \kappa_t$. Then (possibly after further shrinking the neighborhood of $t=0$) p_t will satisfy the assumptions (1.1)–(1.5) and since κ_t are complex canonical transformations, we also know that

$$\omega_t = \omega \text{ is independent of } t. \tag{3.2}$$

In order to prove Theorem 1.2, it suffices to show

Theorem 3.1 *For every neighborhood V of $0 \in \mathbf{C}$, we can find a family G_t as above, such that every neighborhood of $t=0$ will contain a t for which $w_t \not\equiv w$ in V. Here w_t denotes the Weyl density of p_t, defined as in (1.9).*

Remark 3.2 Actually, we shall prove the theorem in all dimensions (replacing 2 by any $0 < n \in \mathbf{N}$) for any $p \not\equiv 0$ that is bounded and holomorphic in a tubular neighborhood of $\mathbf{R}^{2n}$ in $\mathbf{C}^{2n}$ that satisfies (1.2) and for which $p^{-1}(0) \cap \mathbf{R}^{2n} \neq \emptyset$. Let $w_t(z)L(dz)$ be the measure defined as in (1.9) with $p = p_0$ replaced by $p_t = p \circ \kappa_t$.

Proof For $f \in C_0^\infty(V; \mathbf{R})$ we get,

$$\int f(z)\frac{\partial w_t(z)}{\partial t} L(dz) = \frac{d}{dt}\int f(z) w_t(z) L(dz) =$$
$$\frac{d}{dt}\iint f(p_t(x,\xi))dx d\xi = \iint (\frac{\partial f}{\partial z}(p_t)\frac{\partial p_t}{\partial t} + \frac{\partial f}{\partial \overline{z}}(p_t)\frac{\partial \overline{p}_t}{\partial t})dx d\xi.$$

Here, we have

$$\frac{\partial p_t}{\partial t} = iH_{G_t}p_t,$$

and using that f is real,

$$\begin{aligned}
\int f(z)\frac{\partial w_t(z)}{\partial t}L(dz) &= 2\Re(i\iint \frac{\partial f}{\partial z}(p_t)H_{G_t}p_t dx d\xi) \qquad (3.3)\\
&= -2\Re(i\iint \frac{\partial f}{\partial z}(p_t)H_{p_t}(G_t)dx d\xi)\\
&= 2\Re(i\iint H_{p_t}\left(\frac{\partial f}{\partial z}(p_t)\right)G_t dx d\xi)\\
&= 2\Re(i\iint (\frac{\partial^2 f}{\partial z^2}H_{p_t}(p_t) + \frac{\partial^2 f}{\partial \overline{z}\partial z}H_{p_t}(\overline{p}_t)G_t dx d\xi)\\
&= 2\Re(\iint \frac{\partial^2 f}{\partial \overline{z}\partial z}(p_t)i\{p_t, \overline{p}_t\}G_t dx d\xi)\\
&= \iint (\Delta f)(p_t)\{\Re p_t, \Im p_t\}\Re G_t dx d\xi.
\end{aligned}$$

If $\{\Re p, \Im p\} = \frac{i}{2}\{p, \overline{p}\}$ does not vanish identically, there are points arbitrarily close to $p^{-1}(0)$ where it does not vanish and we can choose $f \in C_0^\infty(V; \mathbf{R})$ (where V is any fixed neighborhood of $0 \in \mathbf{C}$) such that $(\Delta f)(p)\{\Re p, \Im p\}$ does not vanish identically. We can then choose $G = G_0$ independent of t with the properties above so that

$$\int f(z)\left(\frac{\partial}{\partial t}\right)_{t=0} w_t(z)L(dz) = \iint (\Delta f)(p)\{\Re p, \Im p\}\Re G dx d\xi \neq 0.$$

We get the conclusion of Theorem 3.1 in this case.

If $\{\Re p, \Im p\} \equiv 0$, we choose G real and independent of t in (3.3) and differentiate that identity once with respect to t at $t = 0$ to get:

$$\begin{aligned}\int f(z) \left(\frac{\partial^2 w_t}{\partial t^2}\right)_{t=0} L(dz) &= \iint (\Delta f)(p) \left(\frac{\partial}{\partial t}\right)_{t=0} (\frac{i}{2}\{p_t, \overline{p_t}\}) G dx d\xi \\ &= \iint (\Delta f)(p) \frac{i}{2}(\{iH_G p, \overline{p}\} + \{p, \overline{iH_G p}\}) G dx d\xi \\ &= -\frac{1}{2} \iint (\Delta f)(p)(H_{\overline{p}} H_p G + H_p H_{\overline{p}} G) G dx d\xi.\end{aligned}$$

Here we integrate by parts and use that $H_p p = 0$, $H_{\overline{p}} p = 0$, to get

$$\int f(z) \left(\frac{\partial^2 w_t}{\partial t^2}\right)_{t=0} L(dz) = \iint (\Delta f)(p) |H_p G|^2 dx d\xi.$$

Again we see that we can find $f \in C_0^\infty(V; \mathbf{R})$ and $G = G_0$ as above, so that the last integral is $\neq 0$. The conclusion in the theorem follows in this case also. $\square$

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